Pythagorean Triples

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Introduction

Pythagorean Triples are not a new thing. Maybe they are new to you. It is likely you are already familiar with the geometric figure from which they are noticed. That geometric figure is a right triangle. In our modern day which includes a plethora of technological devices, one may tend to think we know so much about so many things. Or maybe we simply have access to an electronic world library. Mathematics is no new subject and it is evolving as time marches on.

Have you ever wondered how far we have come as humans with mathematics? Of course it would take a world library to discuss humanity's progress with mathematics. We simply aim to enlighten about the ancient yet relevant mathematical topic of Pythagorean Triples. Your mind probably momentarily reverted to a classroom setting of some type where you learned about the pythagorean Theorem. We need to dig further into history than Pythagoras(l.c. 571- c. 497 BCE) (Mark, 2020) to begin our exploration of Pythagorean Triples.

This paper is written more to introduce and provide a brief explanation of Pythagorean Triples. The history and content related to Pythagorean Triples is far reaching. To encapsulate so much history and relevance in one article is simply not feasible.

We will begin our exploration of pythagorean triples with their history. Before we can start looking into how they came to be, we need to have a starting point for our exploration of Pythagorean Triples.

History

Geometry as we know it has roots stemming from Euclid. Today, we refer to it as Euclidean Geometry. We know from the Plimpton 322, a Babylonian clay tablet, that Geometry and mathematics has been around at least as long as the babylonians. This tablet is written in a sexigesimal number system. The tablet was discovered by an antiquities enthusiast Edgar Banks, who later sold it to George Plimpton. “This mathematical tablet was recovered from an unknown place in the Iraq desert. It can be determined, apparently from its style, that it was written originally sometime around 1800 BCE. It is now located at Columbia University. “ (Casselman)

“The ancient Egyptians knew a lot of geometry, but only as applied methods based on testing and experience. For example, to calculate the area of a circle, they made a square whose sides were eight-ninths the length of the circle's diameter. The area of the square was close enough to the area of the circle that they could not detect any difference. Their method implies that pi has a value of 3.16, slightly off its true value of 3.14, but close enough for simple engineering. Most of what we know about ancient Egyptian mathematics comes from the Rhind Papyrus, discovered in the mid-19th century CE and now kept in the British Museum.

Ancient Babylonians also knew a lot of applied mathematics, including the Pythagorean theorem. Archaeological excavations at Nineveh discovered clay tablets with number triplets satisfying the Pythagorean theorem, such as 3-4-5 or, 5-12-13, and with considerably larger numbers.

In *The Elements*, Euclid collected, organized, and proved geometric ideas that were already used as applied techniques. Except for Euclid and some of his Greek predecessors such as Thales (624-548 BCE), Hippocrates (470-410 BCE), Theaetetus (417-369 BCE), and Eudoxus (408-355 BCE), hardly anyone had tried to figure out why the ideas were true or if they applied in general. Thales even became a celebrity in Egypt because he could see the mathematical principles behind rules for specific problems, then apply the principles to other problems such as determining the height of the pyramids." (Palmer, 2015)

Explanation of Mathematics

Pythagorean Triples are three Positive Integers that satisfy the well known Pythagorean Theorem; $a^2 + b^2 = c^2$. Typically a Pythagorean Triple is expressed in the following manner. (a, b, c) . Where *a,b*, and *c*, are the sides of a right triangle. The sides *a* and *b* are legs of this right triangle, and *c* is the hypotenuse of this right triangle. Side *a* has the smallest length. The following relation is accepted for the triple's listing, $a<b<c$. With this definition one might ask how small or how big can *a*,*b*, and *c* be?

The cardinality of the set of Natural Numbers, by definition, is $\aleph_0$. Which is to say that there is a countably infinite number of Natural numbers. There exists a bijection or, a one to one correspondence between the set of Positive Integers and the set of Natural Numbers. Then, we know that the set of Positive Integers also has a cardinality of $\aleph_0$. This tells us that the set of positive Integers is countably infinite.

The following is a proof from Euclid, which leads us to find out the answer to the previously posed question; How big can a,b, and c be?

Proof:

$$a^2 + b^2 = c^2$$

$$b^2 = c^2 - a^2 = (c + a)(c - a)$$

Rewrite as $\frac{(c+a)}{b} = \frac{b}{(c-a)} \Rightarrow \frac{(c+a)}{b} \in Q$

Let $m, n \in N \ : \ \frac{m}{n} \in Q \Rightarrow \frac{(c+a)}{b} = \frac{m}{n}$

Equating reciprocals to find, $\frac{(c-a)}{b} = \frac{n}{m}$

Solving these two equations by use of Elimination, we find the following relationships.

$\frac{c}{b} = \frac{1}{2}(\frac{m}{n} + \frac{n}{m}) = \frac{m^2+n^2}{2nm}$ and $\frac{a}{b} = \frac{1}{2}(\frac{m}{n} - \frac{n}{m}) = \frac{m^2-n^2}{2nm}$

And finally, $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$. Thus, Euclid's equation to determine pythagorean Triples is as such: ($m^2 - n^2$, $2mn$, $m^2 + n^2$).Where: $m>n>0$. This proof is credited to (Maor, 2007).

If we begin to enumerate the possibilities of Euclid's Pythagorean Triples, we find the following. For each increase in *m* and *n*, the number of possible combinations of Pythagorean Triples increases, That is, when $m = 2$ there is one possible triple. This triple occurs when $m = 2, n = 1 \Rightarrow (3, 4, 5)$ Triple. When $m = 3$, there are now two other values *n* can take on. These values are 1 and 2. Allowing for two more Pythagorean Triples. These two triples are, $(8, 6, 10)$ and $(5, 12, 13)$

For *m*=4, n can take on $m - 1 = 3$ more Pythagorean Triples. These triples are; $(15, 8, 17)$, $(12, 16, 20)$ and $(7, 24, 25)$

If we have a specific *m,* where *m>n>0,* and where *m,n* are Integers,we can represent the total number of possible Pythagorean Triples with a sunnation $\Sigma^{m}{}_{m=2}$ $(m - 1)$. This summation is easily identified when noticing the pattern between the number of triples and *m*. To obtain the number of total possible

Triples, we need to add up every possible triple that each specific *m* generates. Table 1 shows the results from, *m=2* to *m=5*. A '-' is used as a blank to separate the pairs of *m* and *n*. The number of triples for a given *m* is recorded with the first entry of that *m*.

A conditional check is performed in the last column to show the pythagorean theorem holds for the given triple.

# of Triples	*m*	*n*		*a*	*b*	*c*	*a*^2+*b*^2=*c*^2
1	2	1		3	4	5	9+16=25 True
2	3	1		8	6	10	64+36=100 True
-	3	2		5	12	13	25+144=169 True
3	4	1		15	8	17	225+64=289 True
-	4	2		12	16	20	144+256=400 True
-	4	3		7	24	25	49+576+=625 True
4	5	1		24	10	26	576+100=676 True
-	5	2		21	20	29	441+400=841 True
-	5	3		16	30	34	256+900=1156 True
-	5	4		9	40	41	81+1600=1681 True

Table 1

Of course Table 1 is only a subset of the possible Pythagorean Triples. We now ask ourselves, did Euclid capture all possible Pythagorean Triples with his formula? We know from earlier that the set of positive Integers has cardinality of $\aleph_0$. Also, we know that, $\Sigma^{m}{}_{m=2}\ (m-1)$ is an infinite sum, since m is a Positive Integer. The set of integers is closed under addition, which is to say that if $a, b \in \mathbb{Z} \Rightarrow a + b \in \mathbb{Z}$. Then we know the cardinality of the set of all of Euclid's Pythagorean Triples is also countably infinite.

If there were to exist a triple which is not produced by Euclid's equations, then we know his equations do not capture all possible triples. We can show a triple which should have been in the range of values from Table 1, but was not created. Consider the (9,12,15) Triple. 81+144=225. Thus we see there is such a triple which is not generated by Euclid's equations. But wait! (9,12,15), Looks a lot like (3,4,5)! In fact, if we scaled the (3,4,5) Triple by a factor of 3, That is (3*3,3*4,3*5)=3*(3,4,5) we would obtain the (9,12,15) Triple! The (3,4,5) triple was generated by Euclid's equations, with m=2 and n=1. This scaling factor is *k*, where *k* is a positive integer.

What does Euclid's formulation generate? They are named, Primitive Pythagorean Triples or Primitive Triples. We see that a multiple of a Primitive Triple, (9,12,15) was not included in the result of his formulation. We can then say a necessary condition for a Primitive Triple is that the greatest common divisor of a,b and c is one. Stated in other words, a,b, and c are relatively prime.

Is this a sufficient condition of a Primitive Triple? No. If m=2,n=2, we would obtain from Euclid's equations a triple of (0,8,0). Which is not a valid triangle, since we cannot have a triangle with a side length of 0. Which suggests that the gcd(a,b,c)=1, is a necessary, but not a sufficient condition of a primitive triple.

Euclid's formulas for a,b and c are in terms of m and n. What does it mean for m and n if a,b and c are coprime, or in other words, relatively prime? It follows that m and n are also coprime. From Euclid's proof we know,

$a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$, Where: $m>n>0$. The *gcd(a)=gcd(m,n)=1, gcd(b)=gcd(m,n)=1*, and the *gcd(c)=gcd(m,n)=1*, This means the greatest common divisor of a primitive triple is the greatest common divisor of *m* and *n*. Rather than checking the *gcd(a,b,c)=1*, for a primitive triple, we can check the *gcd(m,n)=1* to satisfy this condition.

What are the conditions that make up the infinite set of Primitive Triples? In Figure 1 below, we can recognize yet another pattern that exists for primitive triples. Figure 1 enumerates the pythagorean triples for values of *m* and *n*. The highlighted results in the Figure are Primitive Triples, while the others are a multiple of a Primitive Triple. It is noticed that when *n=m*, no triple results from

Euclid's Equations. Looking at pairs of *n,m* we can discover that a primitive triple occurs only when *n,m* are of opposite parity. That is to say for *n* or *m*, one is even and the other must be odd. Or, that one is odd and the other must be even. A few examples of this in the form (*m,n*) are, (7,2),(8,1) and (10,3). Each a Primitive Triple.

n=	1	2	3	4	5	6	7	8	9
m ↓									
1									
2	(3,4,5)								
3	(8,6,10)	(5,12,13)							
4	(15,8,17)	(12,16,20)	(7,24,25)						
5	(24,10,26)	(21,20,29)	(16,30,34)	(9,40,41)					
6	(35,12,37)	(32,24,40)	(27.36,45)	(20,48,52)	(11,60,61)				
7	(48,14,50)	(45,28,53)	(40,42,58)	(33,56,65)	(24,70,74)	(13,84,85)			
8	(63,16,65)	(60,32,68)	(55,48,73)	(48.64,80)	(39,80,89)	(28,96,100)	(15,112,113)		
9	(80,18,82)	(77,36,85)	(72,54,90)	(65,72,97)	(56,90,106)	(45,108,117)	(32,126,130)	(17,144,145)	
10	(99,20,101)	(96,40,104)	(91,60,109)	(84,80,116)	(75,100,125)	(64,120,136)	(51,140,149)	(36,160,164)	(19,180,181)

Figure 1

Figure 1 Note: These triples are expressed in the form (a,b,c) but not in the form a<b<c. They directly correspond with the *a,b,*and *c* from Euclud's equations. This figure was recreated from (Molony, 2016).

The Euclidean Algorithm was used to determine the gcd(a,b,c) in Figure 1.We will provide one example to clarify this process.

gcd(84,80,116)

gcd(84,80)=4

gcd(4,116))=4

gcd(84,80,116)=4.

As seen in Figure 1, this triple was not highlighted since it is not a primitive triple. There exists a greatest common divisor of *a,b,c* that is not 1. The rest were computed using technology, or readily identified as Non-Primitive since there is some integer *k* that is a divisor of *a*,*b* and *c*.

Primitive Pythagorean Triples are three Positive Integers that satisfy the Pythagorean Theorem; $a^2 + b^2 = c^2$. The values of m and *n*, *in* Euclid's equations must have opposite parity and be coprime. Under these conditions one can generate a Primitive Triple. We know there is an infinite number of possible Primitive Triples. If these conditions are not satisfied then the Triple is called a Non-Primitive Triple.

A non-primitive triple can be easily generated from Euclid's equations with a positive integer scaling constant k. By scaling a Primitive Triple by k, we can generate all other Non-Primitive Triples. Like before, Euclid's equations would become; $a = k(m^2 - n^2)$ *),* $b = k(2nm)$, $c = k(m^2 + n^2)$. Given a Non-PrimitiveTriple,

If we find the gcd(a,b,c)=k, we can scale the Non-Primitive Triple by $\frac{1}{k}$. This gives a Primitive Triple in the form, ($\frac{a}{k}, \frac{b}{k}, \frac{c}{k}$)where *a b* and ,*c* are relatively prime. Then there does not exist any integer d such that d|*a*, and d|*b* and d|*c*. Removing the gcd(a,b,c) from each integer in the Non-Primitive Triple, results in a Primitive Triple. To provide one example of this, looking at Figure 1, the Non-Primitive Triple, (84,80,116). We determine the gcd(a,b,c)=4.This is done by first gcd(84,80)=4, then gcd(4,116)=4, which results with gcd(a,b,c)=4. Thus using a scale factor of $\frac{1}{4}$ we find the primitive triple to be, (21,20,29), but it needs to be arranged so that $a<b<c$. Which is the (20,21,29) triple.

A unique fact about a primitive triple. “The only primitive triple that consists of consecutive integers is 3, 4, 5.”(Schaaf, 1999). There are other patterns or trends that appear among thePythagorean Triples . “One side of every Pythagorean triple is divisible by 3, another by 4, and another by 5. One side may have two of these divisors, as in (8, 15, 17), (7, 24, 25), and (20, 21, 29), or even all three, as in (11, 60, 61). ”(Weisstein, n.d.)

“Most elementary number theory texts prove that all primitive triples (a,b,c) are given by the following: $a = u^2 - v^2$, $b = 2uv$, $c = u^2 + v^2$

where u and v are relatively prime integers, not both odd. Notice that a is a difference of squares, so for it to be prime we need that u and v differ by 1. So

$a = 2v + 1$, $b = 2v^2 + 2v$, and $c = 2v^2 + 2v + 1$.

By Schinzel and Sierpinski's Hypothesis H we then *expect* to see infinitely many triples with two prime entries.” (Caldwell, 2020)

Significance and Application

It is hardly feasible to discuss the significance and application of Pythagorean Triples since they are a result of the Pythagorean Theorem,we have learned that pythagorean triples are three positive integers *a,b* and *c* where $a<b<c$ that satisfy the pythagorean theorem,. It would be more sensible to discuss the significance and application of the Pythagorean Theorem instead. One interesting application for Triple is in Cryptography.

The Pythagorean Theorem in some instances can be cumbersome to calculate without an assistive tool. A benefit of Pythagorean triples which make them so useful comes from the fact they are only integers and always integers. It makes for an almost memorized way to calculate a side of a right triangle.

“The Pythagorean theorem isn't just an intriguing mathematical exercise. It's utilized in a wide range of fields, from construction and manufacturing to navigation.”(Kiger, 2020)

“The right Triangle is a fundamental piece in the history of mathematics. “The Pythagoras’ theorem, arguably the most well-known theorem in the world, has greatly helped mankind to evolve. Its useful right angles are everywhere, whether it is a building, a table, a graph with axes, or the atomic structure of a crystal. It is universally applicable”(Gomes, 2018)

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